

An Introduction to Numerical Modeling of the Atmosphere

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Preface

Numerical modeling is one of four broad approaches to the study of the atmosphere. The others are observational studies of the real atmosphere through field measurements and remote sensing, laboratory studies, and theoretical studies. Each of these four approaches has both strengths and weaknesses. In particular, both numerical modeling and theory involve approximations. In theoretical work, the approximations often involve extreme idealizations, e.g., a dry atmosphere on a beta plane, but on the other hand solutions can sometimes be obtained in closed form with a pencil and paper. In numerical modeling, less idealization is needed, but no closed form solution is possible. In most cases, numerical solutions represent particular cases, as opposed to general relationships. Both theoreticians and numerical modelers make mistakes, from time to time, so both types of work are subject to errors in the old-fashioned human sense.

Perhaps the most serious weakness of numerical modeling, as a research approach, is that it is possible to run a numerical model built by someone else without having the foggiest idea how the model works or what its limitations are. Unfortunately, this kind of thing happens all the time, and the problem is becoming more serious in this era of “community” models with large user groups. One of the purposes of this book is to make it less likely that you, the readers, will use a model without having any understanding of how it works.

This introductory survey of numerical methods in the atmospheric sciences is designed to be a practical, “how-to” course, which also conveys sufficient understanding so that after completing the course students are able to design numerical schemes with useful properties, and to understand the properties of schemes designed by others.

This book is based on my class notes. The first version of the notes, put together in 1991, was heavily based on the class notes developed by Prof. Akio Arakawa at UCLA, as they existed in the early 1970s. Arakawa’s influence is still apparent throughout the book.

The Teaching Assistants for the course have made major improvements in the material and its presentation, in addition to their help with the homework and with questions outside of class. I have learned a lot from them, and also through questions and feedback from the students.

Michelle Beckman, Amy Dykstra, and Val Hisam spent countless hours patiently as-

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