

**An Introduction to
Numerical Modeling of the Atmosphere**

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Contents

Preface	1
1 What this book is about	3
1.1 True stories	3
1.2 Elementary models	4
1.3 Numerical models	4
1.4 Errors	5
1.5 Discretization	6
1.6 Physically based design of mathematical methods	7
1.7 What's the worst that could happen?	9
1.8 The utility of numerical models	10
1.9 Where we are going in this book	11
2 The basic equations in vector form	13
2.1 Introduction	13
2.2 The equation of motion	13
2.2.1 Converting to a rotating frame of reference	13
2.2.2 Forces	16
2.2.3 Apparent gravity	16
2.3 The continuity equation	18
2.4 The mechanical energy equation	18
2.5 The thermodynamic energy equation	20
2.6 Moisture conservation	22
2.7 Total energy conservation	22
2.8 The vertically integrated pressure-gradient force	22
2.9 Segue	23
2.10 Problems	23
3 Finite-difference approximations to derivatives	24
3.1 Finite-difference quotients	24
3.2 A fourth-order-accurate approximation	28
3.3 A systematic approach	30

3.3.1	A family of schemes	30
3.3.2	A generalization for use with nonuniform grids	33
3.3.3	A further generalization to higher-order derivatives	37
3.3.4	Extension to two dimensions	39
3.4	Summary	43
3.5	Problems	43
4	Some time-differencing schemes	45
4.1	Introduction	45
4.2	A family of schemes	46
4.3	Discretization error	47
4.4	Some members of the family	50
4.5	Implicit schemes	54
4.6	Iterative schemes	55
4.7	What's next?	57
4.8	Problems	57
5	The oscillation and decay equations	59
5.1	What a difference an i makes	59
5.2	Computational stability	61
5.3	Time differencing schemes for the oscillation equation	62
5.3.1	The solution of the continuous oscillation equation	62
5.3.2	Amplitude errors and phase errors	62
5.3.3	Non-iterative two-level schemes for the oscillation equation	65
5.3.4	Iterative schemes for the oscillation equation	67
5.3.5	The leapfrog scheme for the oscillation equation	68
5.3.6	Getting off to a good start	70
5.3.7	Ad hoc damping of computational modes in time	72
5.3.8	The stability of the leapfrog scheme for the oscillation equation	73
5.3.9	The second-order Adams-Bashforth Scheme for the oscillation equation	77
5.3.10	A survey of time differencing schemes for the oscillation equation	78
5.4	Schemes for the decay equation	78
5.5	Damped oscillations	84
5.6	Nonlinear damping	84
5.7	Summary	87
5.8	Problems	89
6	Riding along with the air	93
6.1	The Lagrangian form	93
6.2	The advective form	94
6.3	The continuity equation	96
6.4	The flux form	96

6.5	Characteristics	97
6.6	Discussion	100
7	The upstream scheme for advection	101
7.1	From there and then to here and now	101
7.2	The discretization error of the upstream scheme	102
7.3	The domain of dependence	103
7.4	Interpolation and extrapolation	106
7.5	Checking the computational stability of the upstream scheme	107
7.5.1	The direct method	107
7.5.2	The energy method	108
7.5.3	von Neumann's method	109
7.6	Including multiple wave numbers	114
7.7	How periodic boundary conditions come into play	116
7.8	Does the solution improve if we refine the grid?	118
7.9	Summary	120
7.10	Problems	121
8	"Forward-in-time" advection schemes	122
8.1	Accuracy and stability of a family of advection schemes	122
8.2	Matsuno time-differencing with centered space differencing	125
8.3	The Lax-Wendroff scheme	126
8.4	The Takaacs scheme	128
8.5	Implicit schemes for the advection equation	129
8.6	Problems	129
9	Advection in multiple dimensions	130
10	Why be square?	134
10.1	Tiling the plane	134
10.2	More about the Laplacian	136
10.2.1	Approximations to the Laplacian on rectangular grids	137
10.2.2	Integral properties of the Laplacian	138
10.3	Problems	140
11	Finite-volume methods	143
11.1	Godunov schemes	143
11.2	Coordinate-free definitions of operators	143
11.3	How does discrete conservation work?	145
11.4	Finite-difference forms of the gradient, divergence, and curl	146
11.5	Mimetic schemes	146
12	Conservative advection schemes	147

12.1	Continuous advection in one dimension	147
12.2	Conserving mass	148
12.3	Conserving an intensive scalar	149
12.4	An advective form	150
12.5	Conserving a function of an advected scalar	150
12.6	Lots of ways to interpolate	152
12.7	Fixers	155
12.8	A flux form of the upstream scheme	155
12.9	Problems	157
13	Computational dispersion	161
13.1	The case of centered space differencing	161
13.1.1	Second-order schemes	161
13.1.2	The group velocity	164
13.1.3	The analyses of Matsuno and Wurtele	168
13.1.4	Fourth-order schemes	172
13.2	Space-uncentered schemes	172
13.3	Even- and odd-order schemes	174
13.4	Summary	176
14	Modern Eulerian advection schemes	177
14.1	Sign-preservation and monotonicity	177
14.2	Hole filling	179
14.3	Flux-corrected transport	181
14.4	TVD schemes	185
14.5	The piecewise parabolic method	185
14.6	van Leer schemes	186
14.7	Leonard schemes	186
14.8	WENO schemes	186
14.9	Prather's scheme	187
15	Lagrangian and semi-Lagrangian advection schemes	188
15.1	Lagrangian schemes	188
15.1.1	Smoothed particle hydrodynamics	189
15.1.2	Slippery sacks	190
15.2	Semi-Lagrangian schemes	191
15.2.1	Look further upstream	191
15.2.2	More accurate semi-Lagrangian schemes	193
15.2.3	Remapping schemes	195
16	Just relax	196
16.1	Introduction	196
16.2	The Poisson equation	197

16.3	A continuous one-dimensional boundary-value problem	198
16.4	Fourier methods to solve the Poisson equation	199
16.5	Finite-difference methods to solve the Poisson equation	199
16.6	Jacobi relaxation	201
16.7	Gauss-Seidel relaxation	206
16.8	The alternating-direction implicit method	208
16.9	Multigrid methods	209
16.10	Summary	213
16.11	Problems	214
17	It's Only Dissipation (But I Like It)	217
17.1	The diffusion equation	217
17.2	A simple explicit scheme	219
17.3	An implicit scheme	221
17.4	The DuFort-Frankel scheme	223
17.5	Hyperdiffusion	224
17.6	Summary	225
17.7	Problems	225
18	The shallow-water equations	226
18.1	Introduction	226
18.2	Energy conservation in shallow water	230
18.2.1	Potential enstrophy conservation	231
18.3	The nondivergent barotropic vorticity equation	231
19	Making waves	233
19.1	Waves in shallow water	233
19.2	Pure gravity waves on one-dimensional grids	236
19.3	Inertia-gravity waves on two-dimensional staggered grids	241
19.4	Dependence on the radius of deformation	247
19.5	Other meshes	249
19.6	Time-differencing schemes for the shallow-water equations	251
19.6.1	Centered in space and time	251
19.6.2	Implicit schemes	254
19.6.3	The forward-backward scheme	257
19.7	Summary and conclusions	259
19.8	Problems	259
20	Something there is that doesn't love a wall	261
20.1	Introduction	261
20.2	Real walls	261
20.3	Advection at inflow boundaries	263
20.4	Advection at outflow boundaries	271

20.5	Energy fluxes at outflow boundaries	276
20.6	Advection on inhomogeneous grids	280
20.6.1	What does the downstream signal look like?	282
20.6.2	Reflection and transmission	285
20.6.3	Choosing the weights at a seam	286
20.7	Boundary conditions for waves	287
20.8	The effects of a mean flow	291
20.9	Summary	292
20.10	Problems	292
21	The sound of silence	294
21.1	Sound waves	294
21.2	Listening for sound waves	294
21.3	Coping with acoustic waves	296
21.4	Filtering sound waves	296
21.4.1	The anelastic and Boussinesq systems	296
21.4.2	The quasi-static system	298
21.5	The Unified System	299
21.6	Dispersion curves for the various systems of equations	301
22	Stairways to heaven	303
22.1	The third dimension is special	303
22.2	Choosing a vertical coordinate system	304
22.3	The basic equations in height coordinates	305
22.4	Transformation to a general vertical coordinate	306
22.4.1	Up is up	306
22.4.2	The pseudodensity	306
22.5	ALE	309
22.6	Boundary conditions on the continuity equation	310
22.7	The vertically integrated horizontal pressure-gradient force	311
22.8	Energy conservation with generalized vertical coordinates	312
22.9	The energy equation with the quasi-static approximation	316
22.10	The potential vorticity equation	318
22.11	Segue	318
23	A survey of vertical coordinates	319
23.1	Height coordinates	320
23.2	Pressure and log pressure coordinates	325
23.2.1	The hydrostatic pressure	325
23.2.2	The hydrostatic pressure as a vertical coordinate	326
23.3	The sigma coordinate	329
23.3.1	Definition	329
23.3.2	The continuity equation in sigma coordinates	331

23.3.3	The horizontal pressure-gradient force	332
23.3.4	The vertically integrated horizontal pressure-gradient force	334
23.3.5	Energy conversion	334
23.4	Hybrid sigma-pressure coordinates	335
23.5	Terrain-following vertical coordinates based on height	335
23.6	The eta-coordinate	336
23.7	Potential temperature and entropy	338
23.8	Vertical mass flux for a family of vertical coordinates with the quasi-static approximation	343
23.9	Hybrid sigma-theta coordinates	347
23.10	Summary of vertical coordinate systems	349
23.11	Problems	349
24	Vertical differencing	352
24.1	Vertical staggering	352
24.2	Conservation of total energy with continuous sigma coordinates	354
24.3	Conservation properties of vertically discrete models using sigma-coordinates	358
24.3.1	The horizontal pressure-gradient force	360
24.3.2	The thermodynamic energy equation	361
24.3.3	The mechanical energy equation	363
24.3.4	Total energy conservation	364
24.3.5	The problem with the L grid	366
24.4	Summary and conclusions	368
24.5	Problems	369
25	Conservative schemes for the one-dimensional nonlinear shallow-water equations	370
25.1	Properties of the continuous equations	370
25.2	The spatially discrete case	373
25.3	Summary	381
25.4	Problems	381
26	When the advector is the advectee	382
26.1	Introduction	382
26.2	Scale interactions and nonlinearity	382
26.2.1	Aliasing error	383
26.2.2	Almost famous	383
26.2.3	A mathematical view of aliasing	384
26.3	Advection by a variable, non-divergent current	387
26.4	Aliasing instability	390
26.4.1	An example of aliasing instability	391
26.4.2	Analysis in terms of discretization error	395
26.4.3	Discussion	397

26.5	Fjortoft's Theorem	398
26.6	Kinetic energy and enstrophy conservation in two-dimensional non-divergent flow	404
26.7	The effects of time differencing on conservation of squares	418
26.8	Conservative schemes for the two-dimensional shallow water equations with rotation	420
26.9	Angular momentum conservation	423
26.10	Summary	424
26.11	Problems	424
27	Finite differences on the sphere	427
27.1	Introduction	427
27.2	Spherical coordinates	428
27.2.1	Vector calculus in spherical coordinates	428
27.2.2	The "pole problem"	429
27.2.3	Polar filters	434
27.3	The Kurihara grid	436
27.4	Displaced poles	437
27.5	Grids Based on Map Projections	437
27.6	Composite grids	441
27.7	Unstructured spherical grids	442
27.8	Summary	448
27.9	Problems	448
28	Spectral methods	449
28.1	Introduction	449
28.2	Transform pairs	449
28.3	Differentiation	450
28.4	Truncation	450
28.5	Spectral differentiation in terms of finite-differences	453
28.6	Solving linear equations with the spectral method	454
28.7	Solving nonlinear equations with the spectral method	456
28.8	The transform method	457
28.9	Spectral methods on the sphere	459
28.9.1	Spherical harmonics	459
28.9.2	Truncation	461
28.10	Spherical harmonic transforms	463
28.11	How it works	464
28.12	Semi-implicit time differencing	465
28.13	Conservation properties and computational stability	466
28.14	The accuracy of spectral models	466
28.15	Physical parameterizations	468

28.16 Moisture advection	468
28.17 Linear grids	469
28.18 Reduced linear grids	469
28.19 Summary	469
28.20 Problems	470
29 Finite-Element Methods	472
29.1 Problems	473
30 Concluding discussion	474
Appendices	475
A Vectors, Coordinates, and Coordinate Transformations	475
A.1 Physical laws and coordinate systems	475
A.2 Scalars, vectors, and tensors	475
A.3 Differential operators	478
A.4 Vector identities	480
A.5 Spherical coordinates	482
A.5.1 Vector operators in spherical coordinates	482
A.5.2 Horizontal and vertical vectors in spherical coordinates	483
A.5.3 Derivation of the gradient operator in spherical coordinates	485
A.5.4 Applying vector operators to the unit vectors in spherical coordinates	486
A.6 Solid body rotation	487
A.7 Formulas that are useful for two-dimensional flow	488
A.8 Vertical coordinate transformations	489
A.9 Concluding summary	492
B A Demonstration that the Fourth-Order Runge-Kutta Scheme Really Does Have Fourth-Order Accuracy	493
C Spherical Harmonics	502
Bibliography	513

Preface

Numerical modeling is one of four broadly defined approaches to the study of the atmosphere. The others are observational studies of the real atmosphere through field measurements and remote sensing, laboratory studies, and theoretical studies. Each of these approaches has both strengths and weaknesses. In particular, both numerical modeling and theory involve approximations. In theoretical work, the approximations often involve extreme idealizations, e.g., a dry atmosphere on a beta plane, but on the other hand solutions can sometimes be obtained in closed form with a pencil and paper. In numerical modeling, less idealization is needed, but no closed form solution is possible. In most cases, numerical solutions represent particular cases, as opposed to general relationships. Both theoreticians and numerical modelers make mistakes, from time to time, so both types of work are subject to errors in the old-fashioned human sense.

Perhaps the most serious weakness of numerical modeling, as a research approach, is that it is possible to run a numerical model built by someone else without having the foggiest idea how the model works or what its limitations are. Unfortunately, this kind of thing happens all the time, and the problem is becoming more serious in this era of “community” models with large user groups. One of the purposes of this book is to make it less likely that you, the readers, will use a model without having any understanding of how it works.

This introductory survey of numerical methods in the atmospheric sciences is designed to be a practical, “how-to” course, which also conveys sufficient understanding so that after completing the course students are able to design numerical schemes with useful properties, and to understand the properties of schemes designed by others.

This book is based on my class notes. The first version of the notes, put together in 1991, was heavily based on the class notes developed by Prof. Akio Arakawa at UCLA, as they existed in the early 1970s. Arakawa’s influence is still apparent throughout the book.

The Teaching Assistants for the course have made major improvements in the material and its presentation, in addition to their help with the homework and with questions outside of class. I have learned a lot from them, and also through questions and feedback from the students.

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